JOURNAL OF APPROXIMATION THEORY 27, 135–145 (1979)

Bounds for the Norm of Certain Spline Projections*

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Communicated by E. W. Cheney

Received February 17, 1978

1. INTRODUCTION AND NOTATION

Let n and q be given natural numbers such that $n+1 \geqslant q > 0$ $(n > 0)$. Further, let $I = [0, 1]$, and let Δ denote an arbitrary but fixed partition of the interval I: $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. By $\mathrm{Sp}(2q-1, \Delta)$ we denote the space of *natural spline functions of degree* $2q-1$; thus $s \in \mathrm{Sp}(2q-1, \Delta)$ iff:

(i) in each interval $[x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$ s coincides with an algebraic polynomial of degree at most $2q-1$,

(ii) $s \in C^{2q-2}[0, 1]$,

(iii) $s^{(j)}(0) = s^{(j)}(1) = 0$ $(j = q, q+1, \ldots, 2q-2)$.

It is known (see, e.g., [1]) that for given real numbers f_i $(i = 0, 1, \ldots, n)$ there exists exactly one $s \in \mathrm{Sp}(2q-1, \Delta)$ interpolating the data f_i:

$$s(x_i) = f_i \qquad (i = 0, 1, \ldots, n) \tag{1.1}$$

(we may assume that $f_i = f(x_i)$, where $f \in C[0, 1]$). Every such spline function may be written in the following way:

$$s(x) = \sum_{i=0}^{n} f_i s_i(x) \qquad (x \in I),$$

where $s_i \in \mathrm{Sp}(2q-1, \Delta)$, $s_i(x_j) = \delta_{ij}$ $(i, j = 0, 1, \ldots, n)$. The functions s_i are the so-called *fundamental spline functions*. Consider the operator L_n^{2q-1} defined by

$$L_n^{2q-1} f(x) = \sum_{i=0}^{n} f(x_i)\, s_i(x) \qquad (f \in C[0, 1]). \tag{1.2}$$

* This paper was completed while the author was visiting at the State University of New York at Stony Brook.

0021-9045/79-100135-11$02.00/0

It is obvious that L_n^{2q-1} is a linear, bounded, and idempotent operator with domain $C[0, 1]$ and range $\mathrm{Sp}(2q-1, \Delta)$; thus L_n^{2q-1} is a *projection.* We have the elementary but important inequality

$$\|f - L_n^{2q-1}f\|_\infty \leqslant (1 + \| L_n^{2q-1} \|) \operatorname{dist}(f, \mathrm{Sp}(2q-1, \Delta)),$$

where $\| \cdot \|_\infty$ stands for the sup-norm in the interval I, and

$$\| L_n^{2q-1} \| = \sup_{\|f\|_\infty \leqslant 1} \| L_n^{2q-1} f\|_\infty \qquad (f \in C[0, 1]).$$

From the above inequality it follows that the information on the size of the norm of the projection L_n^{2q-1} is important. Some results on the norm of the above projection are known in the periodic case, i.e., when conditions (iii) are changed by the following $s^{(j)}0) = s^{(j)}(1)$ $(j = 0, 1,..., 2q-2)$, but the function f in (1.2) is a periodic function such that $f(0) = f(1)$ (see [5], [10–11], [13–17]).

In Section 2 some lemmas are given. In Section 3 the cubic case ($q = 2$) is investigated. Estimations from above for $\| L_n{}^3 \|$ (for arbitrary knots), and from below for $\| L_n{}^3 \|$ (for equidistant knots) are given. In the final section a theorem is given in which the quantity $\| L_n{}^5 \|$ is estimated from above (in the case of equidistant knots).

2. Some Lemmas

We define the sequence $\{d_i\}$ in the following way: $d_{-1} = 0$, $d_0 = 1$, $d_{i+1} = 4d_i - d_{i-1}$ $(i = 0, 1,...)$.

Lemma 2.1. *For the numbers d_i defined as above the following inequalities hold*:

$$(2 + 3^{1/2})\, d_i < d_{i+1} \leqslant 4d_i \qquad (i = 0, 1,...). \tag{2.1}$$

Proof. Solving the above difference equation we obtain

$$d_i = [(3 - 2(3)^{1/2})(2 - 3^{1/2})^i + (3 + 2(3)^{1/2})(2 + 3^{1/2})^i]/6 \equiv a_i + b_i\,,$$

where $a_i = (3 - 2(3)^{1/2})(2 - 3^{1/2})^i/6$. Hence

$$\begin{aligned} d_{i+1} &= a_i(2 - 3^{1/2}) + b_i(2 + 3^{1/2}) = 2d_i + 3^{1/2}(d_i - 2a_i) \\ &= (2 + 3^{1/2})\, d_i - 2(3)^{1/2}\, a_i > (2 + 3^{1/2} d_i), \end{aligned}$$

since $a_i < 0$. The second inequality in (2.1) is obvious. ∎

Let $\beta_{j,-1} = \beta_{j0} = \beta_{jn} = \beta_{j,n+1} = 0$, and

$$\begin{aligned}\beta_{ij} &= (-1)^{j+i}\, d_{j-1}d_{n-i-1}/d_{n-1} \qquad (j \leqslant i),\\ &= (-1)^{j+i}\, d_{i-1}d_{n-j-1}/d_{n-1} \qquad (j \geqslant i),\\ &\qquad\qquad (i, j = 1, 2,..., n-1). \end{aligned} \tag{2.2}$$

LEMMA 2.2. *If the numbers $m_j^{(i)}$ are such that*

$$\begin{aligned} &m_{j-1}^{(i)} + 4m_j^{(i)} + m_{j+1}^{(i)} = 6n^2(\delta_{j+1,i} - \delta_{ji} + \delta_{j-1,i}),\\ &m_0^{(i)} = m_n^{(i)} = 0 \qquad (i = 0, 1,..., n;\ j = 1, 2,..., n-1), \end{aligned} \tag{2.3}$$

then

$$\begin{aligned} m_j^{(i)} &= (-1)^{j+1}\, 6n^2 d_{n-j-1}/d_{n-1} \qquad (i = 0),\\ &= (-1)^{j+i+1}\, 36n^2 d_{j-1}d_{n-i-1}/d_{n-1} \qquad (j < i),\\ &= -6n^2(d_{i-2}d_{n-i-1} + 2d_{i-1}d_{n-i-1} + d_{i-1}d_{n-i-2})/d_{n-1}\\ &\qquad\qquad (j = i;\ i = 1, 2,..., n-1),\\ &= (-1)^{j+i+1}\, 36n^2 d_{i-1}d_{n-j-1}/d_{n-1} \qquad (j > i),\\ &= m_{n-j}^{(i)} \qquad (i = n). \end{aligned} \tag{2.4}$$

Proof. It is known (see, e.g., [12]) that a matrix of the above system of linear equations (2.3) possesses an inverse matrix with entries given by (2.2). Hence, and from (2.3), we obtain

$$m_j^{(i)} = 6n^2(\beta_{j,i-1} - \beta_{ji} + \beta_{j,i+1})$$

and further, in virtue of (2.2), we obtain (2.4). ∎

LEMMA 2.3. *Let the knots x_i be equidistant ($x_i = i/n$; $i = 0, 1,..., n$). If $q = 2$ and $x \in [x_{j-1}, x_j]$ ($j = 1, 2,..., n$) then*

$$\begin{aligned} \operatorname{sgn} s_i(x) &= (-1)^{i+j} \qquad (j \leqslant i),\\ &= (-1)^{i+j+1} \qquad (j > i),\\ &\qquad (i = 0, 1,..., n;\ j = 1, 2,..., n). \end{aligned} \tag{2.5}$$

Proof. If $x \in [x_{j-1}, x_j]$ then the fundamental cubic spline function $s_i(x)$ may be written in the following way:

$$\begin{aligned} s_i(x) &= \delta_{j-1,i}(1-t) + \delta_{ij}t\\ &\quad + \{m_{j-1}^{(i)}[(1-t)^3 - (1-t)] + m_j^{(i)}(t^3 - t)\}/6n^2, \end{aligned} \tag{2.6}$$

where $t = n(x - x_{j-1})$. The proof of the above equality in (2.5) will be divided into three cases.

Case 1°. $|i-j| > 1$. Let $i = 0$. From (2.6) and (2.4) we have

$$s_0(x) = (-1)^j[(t-2)\,d_{n-j} + (t+1)\,d_{n-j-1}]\,t(1-t)/d_{n-1}\,.$$

By virtue of Lemma 2.1 it follows that the expression in the square brackets is negative for $0 \leqslant t \leqslant 1$. Hence $\operatorname{sgn} s_0(x) = (-1)^{j+1}$ for $x \in [x_{j-1}, x_j]$. Quite similarly we can prove (2.5) for $i > 0$.

Case 2°. $j = i + 1$ ($i = 0, 1,..., n-1$). By virtue of (2.6) we have the following expression for the fundamental spline function $s_i(x)$ ($x \in [x_{j-1}, x_j]$):

$$s_i(x) = (1-t)\{1 + [m_i^{(i)}(t^2 - 2t) - m_{i+1}^{(i)}(t^2 + t)]/6n^2\}.$$

From (2.4) it follows that $m_i^{(i)} < 0$, $m_{i+1}^{(i)} > 0$. Hence a coefficient before t^3 in the last expression is a positive. For $i > 0$ $s_i(x)$ vanish in x_{i-1}, x_{i+1}, and on the right of x_{i+1}. Thus $s_i(x) > 0$ for $x \in [x_i, x_{i+1})$.

Case 3°. $j = i$ ($i = 1, 2,..., n$). In this case we have

$$s_i(x) = t\{1 + [m_{i-1}^{(i)}(-t^2 + 3t - 2) + m_i^{(i)}(t^2 - 1)]/6n^2\}.$$

Let $i = 2, 3,..., n-1$. From (2.4) we obain $m_{i-1}^{(i)} > 0$, $m_i^{(i)} < 0$. Hence $s_i(x)$ vanish in the points x_{i+1}, x_{i-1} and on the left of x_{i-1}. Finally $s_i(x) > 0$ if $x \in (x_{i-1}, x_i]$. Similarly we can prove that $s_1(x) > 0$ if $x \in (x_0, x_1]$ and $s_n(x) > 0$ if $x \in (x_{n-1}, x_n]$. ∎

3. Cubic Case

Now we introduce some additional notation. Let $h_j = x_j - x_{j-1}$ ($j = 1, 2,..., n$), $h = \max_{1 \leqslant j \leqslant n} h_j$, $\mathbf{h} = \min_{1 \leqslant j \leqslant n} h_j$, $M_n = h/\mathbf{h}$, $\lambda_j = h_{j+1}/(h_j + h_{j+1})$, $\mu_j = 1 - \lambda_j$ ($j = 1, 2,..., n-1$), $m_j = s''(x_j)$ ($j = 0, 1,..., n$), where $s \in \mathrm{Sp}(3, \Delta)$.

The following theorem holds

THEOREM 3.1. *For arbitrary knots x_i ($i = 0, 1,..., n$),*

$$\| L_n^3 \| \leqslant 1 + \tfrac{3}{2} M_n^2.$$

Proof. The above defined numbers m_j satisfied the so-called *consistency relations* (see, e.g., [1])

$$\mu_j m_{j-1} + 2m_j + \lambda_j m_{j+1} = \frac{6}{h_j + h_{j+1}} \left(\frac{f_{j+1} - f_j}{h_{j+1}} - \frac{f_j - f_{j-1}}{h_j} \right)$$

$$(j = 1, 2, \ldots, n-1;\ m_0 = m_n = 0). \quad (3.1)$$

Using a standard diagonal dominance argument to the above system (3.1) we obtain

$$\max_{1 \leqslant j \leqslant n-1} |m_j| \leqslant 6\omega(f, h)/\mathbf{h}^2, \quad (3.2)$$

where $\omega(\cdot, \cdot)$ denotes usual mudulus of continuity. For $x \in [x_{j-1}, x_j]$ the spline function $s(x)$ has the form

$$s(x) = f_{j-1}(1 - t) + f_j t + \frac{h_j^2}{6} \{m_{j-1}[(1-t)^3 - (1-t)] + m_j(t^3 - t)\},$$

where $t = (x - x_{j-1})/h_j$. Hence, and from (3.2), we obtain

$$|s(x)| \leqslant \|f\|_\infty + \tfrac{3}{4} M_n^2 \omega(f, h). \quad (3.3)$$

For the function $f \in C[0, 1]$, and such that $\|f\|_\infty \leqslant 1$, the obvious inequality $\omega(f, h) \leqslant 2$ holds. Hence, and from (3.3), we obtain the desired inequality in the thesis of the above theorem. ∎

COROLLARY 3.1. *For equidistant knots we have* $\| L_n^3 \| \leqslant \frac{5}{2}$.

Now some estimations from below for $\| L_n^3 \|$ will be given in the case of equidistant knots. Let

$$\Lambda_n^{2q-1}(x) \equiv \sum_{l=0}^{n} |s_l(x)| \qquad (x \in I)$$

denote the so-called *Lebesgue function* for the projection L_n^{2q-1}. It is known that $\| L_n^{2q-1} \| = \| \Lambda_n^{2q-1} \|_\infty$. Now we give the explicit form for the function $\Lambda_n^3(x)$ $(x \in I)$ in the case when knots x_i are equidistant. By virtue of (2.6) and (2.5) we have for $x \in [x_{i-1}, x_i]$,

$$\Lambda_n^3(x) = \sum_{l=0}^{n} |s_l(x)| = \sum_{l=0}^{i-1} (-1)^{i+l+1} s_l(x) + \sum_{l=i}^{n} (-1)^{i+l} s_l(x)$$

$$= 1 + \sum_{l=0}^{i-1} (-1)^{i+l+1} [m_{i-1}^{(l)} C_i(x) + m_i^{(l)} D_i(x)]$$

$$+ \sum_{l=i}^{n} (-1)^{i+l} [m_{i-1}^{(l)} C_i(x) + m_i^{(l)} D_i(x)].$$

where

$$C_i(x) = [(1-t)^3 - (1-t)]/6n^2, \qquad D_i(x) = (t^3 - t)/6n^2$$
$$(t = n(x - x_{i-1});\ i = 1, 2, \ldots, n). \quad (3.4)$$

For $x \in [x_{i-1}, x_i]$ we have $C_i(x) \leqslant 0$, $D_i(x) \leqslant 0$. Let

$$\begin{aligned} \alpha_{i,n} &= \sum_{l=0}^{i-1} (-1)^{i+l+1} m_{i-1}^{(l)} + \sum_{l=i}^{n} (-1)^{i+l} m_{i-1}^{(l)}, \\ \beta_{i,n} &= \sum_{l=0}^{i-1} (-1)^{i+l+1} m_i^{(l)} + \sum_{l=i}^{n} (-1)^{i+l} m_i^{(l)}. \end{aligned} \quad (3.5)$$

The Lebesgue function $\Lambda_n^3(x)$ may be written in terms $\alpha_{i,n}$ and $\beta_{i,n}$ in the following way:

$$\Lambda_n^3(x) = 1 + \alpha_{i,n} C_i(x) + \beta_{i,n} D_i(x) \qquad (x \in [x_{i-1}, x_i];\ i = 1, 2, \ldots, n). \quad (3.6)$$

The numbers $\alpha_{i,n}$ and $\beta_{i,n}$ may be expressed by the numbers d_k. Thus by virtue of (2.4) we have

$$\begin{aligned} \alpha_{i,n} &= -\frac{6n^2}{d_{n-1}} \left[\left(1 + 6 \sum_{l=0}^{i-3} d_l\right) d_{n-i} + d_{i-3} d_{n-i} + 2 d_{i-2} d_{n-i} + d_{i-2} d_{n-i-1} \right. \\ &\qquad \left. - \left(1 + 6 \sum_{l=0}^{n-i-1} d_l\right) d_{i-2} \right], \\ \beta_{i,n} &= -\frac{6n^2}{d_{n-1}} \left[\left(1 + 6 \sum_{l=0}^{n-i-2} d_l\right) d_{i-1} + d_{i-2} d_{n-i-1} + 2 d_{i-1} d_{n-i-1} \right. \\ &\qquad \left. + d_{i-1} d_{n-i-2} - \left(1 + 6 \sum_{l=0}^{i-2} d_l\right) d_{n-i-1} \right] \qquad (i = 1, 2, \ldots, n). \end{aligned} \quad (3.7)$$

THEOREM 3.2. *Let $x_i = i/n$ $(i = 0, 1, \ldots, n)$. Then*

$$\begin{aligned} \| L_n^3 \| &\geqslant \gamma_n \quad \text{for} \quad n = 2m + 1, \\ &\geqslant \delta_n \quad \text{for} \quad n = 2m, \end{aligned} \qquad (m = 1, 2, \ldots),$$

where

$$\gamma_n = 1 + \frac{3}{4 d_{n-1}} (d_{2j-1+k} - d_{2j-2+k}) \left(6 \sum_{l=0}^{2j-3-k} d_l + d_{2j-2+k} + d_{2j-3+k} + 1\right),$$
$$\begin{aligned} k &= 0 \quad \text{for} \quad n = 4j - 1 \quad (j = 1, 2, \ldots), \\ &= -1 \quad \text{for} \quad n = 4j - 3 \quad (j = 2, 3, \ldots), \end{aligned}$$

$$\delta_n = 1 + \frac{3}{8d_{n-1}}$$
$$\times \left[(d_{2j+k} - d_{2j-2+k}) \left(1 + 6 \sum_{l=0}^{2j-2+k} d_l\right) - 2d_{2j-1+k}(d_{2j-1+k} + d_{2j-2+k}) \right],$$
$$\begin{aligned} k &= 0 \qquad \textit{for} \quad n = 4j, \\ &= -1 \qquad \textit{for} \quad n = 4j-2. \end{aligned} \qquad (j = 1, 2, \ldots).$$

Additionally $\| L_1^3 \| = 1$.

Proof. From (3.4) and (3.6)–(3.7) it follows that $\Lambda_n^3(x) = \Lambda_n^3(1-x)$ $(x \in I)$. Thus investigation of the function $\Lambda_n^3(x)$ may be done only for $x \in [0, 1/2]$.

Assume n is odd. Let

1°. $n = 4j - 1$ $(j = 1, 2, \ldots)$. Then putting $i = 2j$ in (3.6) and (3.7), we obtain

$$\alpha_{2j,4j-1} = -\frac{6n^2}{d_{n-1}} \left[\left(1 + 6 \sum_{l=0}^{2j-3} d_l\right) d_{2j-1} + d_{2j-3}d_{2j-1} + 2d_{2j-2}d_{2j-1} + d_{2j-2}^2 \right.$$
$$\left. - \left(1 + 6 \sum_{l=0}^{2j-3} d_l\right) d_{2j-2} \right].$$

Using the obvious equality $d_{2j-3}d_{2j-1} + 2d_{2j-2}d_{2j-1} - 5d_{2j-2}^2 = (d_{2j-1} - d_{2j-2})(d_{2j-2} + d_{2j-3})$ we obtain finally

$$\alpha_{2j,4j-1} = -\frac{6n^2}{d_{n-1}} (d_{2j-1} - d_{2j-2}) \left(6 \sum_{l=0}^{2j-3} d_l + d_{2j-2} + d_{2j-3} + 1\right) < 0,$$
$$\beta_{2j,4j-1} = \alpha_{2j,4j-1}\,. \tag{3.8}$$

From (3.6), (3.4), and (3.8) it follows that the function $\Lambda_n^3(x)$ is strictly concave in the interval (x_{2j-1}, x_{2j}), and hence $\max_{x_{2j-1} \leqslant x \leqslant x_{2j}} \Lambda_n^3(x) = \Lambda_n^3(1/2) = \gamma_n$.

2°. $n = 4j - 3$ $(j = 2, 3, \ldots)$. In this case we put $i = 2j - 1$. Similarly calculations as above give the desired result. For $n = 1$ from (2.3) and (3.5)–(3.6) it follows that $\Lambda_1^3(x) \equiv 1$. Hence $\| L_1^3 \| = 1$.

Assume n is even. Let

3°. $n = 4j - 2$ $(j = 1, 2, \ldots)$. Putting $i = 2j - 1$ in (3.6) we obtain by virtue of (3.7)

$$\alpha_{2j-1,4j-2} = -\frac{6n^2}{d_{n-1}}(d_{2j-1} - d_{2j-3})\left(1 + 3\sum_{l=0}^{2j-3} d_l\right) - 2\beta_{2j-1,4j-2},$$

$$\beta_{2j-1,4j-2} = -\frac{12n^2}{d_{n-1}} d_{2j-2}(d_{2j-2} + d_{2j-3}) < 0.$$

Now we can prove that $\alpha_{2j-1,4j-2} \leqslant 0$. The equivalent inequality to the above is the following:

$$(d_{2j-1} - d_{2j-3})\left(1 + 6\sum_{l=0}^{2j-3} d_l\right) \geqslant 4d_{2j-2}(d_{2j-2} + d_{2j-3}).$$

Let L denote the left hand of the above inequality. Further we have

$$L = 2(2d_{2j-2} - d_{2j-3})\left(1 + 6\sum_{l=0}^{2j-3} d_l\right) > 12d_{2j-3}\left(1 + 6\sum_{l=0}^{2j-3} d_l\right).$$

The last inequality follows from the inequality $d_{2j-2} > 3.5d_{2j-3}$ (see Lemma 2.1). Further, by virtue of $4d_{i-1} \geqslant d_i$, we obtain

$$L > 3d_{2j-2}\left(1 + 6\sum_{l=0}^{2j-3} d_l\right) = 4d_{2j-2}\left(.75 + 4.5\sum_{l=0}^{2j-3} d_l\right)$$
$$> 4d_{2j-2}(d_{2j-3} + d_{2j-2}).$$

Thus the function $\Lambda_n^3(x)$ is strictly concave in the interval (x_{2j-2}, x_{2j-1}). Putting $\delta_n = \Lambda_n^3(1/2(x_{2j-2} + x_{2j-1}))$ we obtain the desired result.

4°. $n = 4j$ ($j = 1, 2, \ldots$). In this case we take $i = 2j$, and define $\delta_n = \Lambda_n^3(1/2(x_{2j-1} + x_{2j}))$. ∎

Now we give some numerical values for the quantities γ_n and δ_n for small values of n:

$$\gamma_3 = 1\,\frac{3}{10} = 1.3, \qquad \gamma_5 = 1\,\frac{9}{19} = 1.4736\ldots,$$

$$\gamma_7 = 1\,\frac{75}{142} = 1.5281\ldots, \qquad \gamma_9 = 1\,\frac{2448}{4505} = 1.5433\ldots,$$

$$\delta_2 = 1\,\frac{3}{16} = 1.1875, \qquad \delta_4 = 1\,\frac{29}{68} = 1.3883\ldots,$$

$$\delta_6 = 1\,\frac{521}{1040} = 1.5009\ldots, \qquad \delta_8 = 1\,\frac{23283}{43546} = 1.5357\ldots.$$

Conjecture. For all odd n ($n > 3$) $\gamma_n = \| L_n^3 \|$. For all natural n ($n > 0$) $\| L_n^3 \| < (1 + 3(3^{1/2})/4 = 1.5490\ldots$.

4. Quintic Case

Now we assume that the knots x_i are equidistant. Let $s_i^{(j)} = s^{(j)}(x_i)$ $(i = 0, 1,\dots, n;\ j = 0, 1, 2, 3, 4)$, where $s \in \mathrm{Sp}(5, \Delta)$. First we prove the following

LEMMA 4.1. *For the equidistant knots x_i the following estimations hold*:

$$\max_{0\leqslant i\leqslant n} |s_i'| \leqslant \frac{23}{3}\, n\omega\left(f, \frac{1}{n}\right), \qquad \max_{0\leqslant i\leqslant n} |s_i''| \leqslant \frac{34}{3}\, n^2\omega\left(f, \frac{1}{n}\right),$$

$$\max_{0\leqslant i\leqslant n} |s_i'''| \leqslant 40n^3\omega\left(f, \frac{1}{n}\right), \qquad \max_{0\leqslant i\leqslant n} |s_i^{\mathrm{IV}}| \leqslant 80n^4\omega\left(f, \frac{1}{n}\right).$$

Proof. Let $A = (a_{ij})$ be a symmetric and five-diagonal matrix $(n-2) \times (n-2)$ and such that $a_{ii} = 66$ $(i = 1, 2,\dots, n-2)$, $a_{i,i+1} = 26$ $(i = 1, 2,\dots, n-3)$, $a_{i,i+2} = 1$ $(i = 1, 2,\dots, n-4)$, $a_{ij} = 0$ for $|i-j| > 2$ $(i, j = 1, 2,\dots, n-2)$. Further, let the numbers γ_j $(j = 1, 2,\dots, n-2)$ be the solution of the following system of linear equations:

$$\sum_{j=1}^{n-2} a_{ij}\gamma_j = 2n^3\Delta^3 f_{i-1} \qquad (i = 1, 2,\dots, n-2), \tag{4.1}$$

where $f_i = s(x_i)$ $(i = 0, 1,\dots, n)$. Using the standard diagonal dominance argument we obtain $\| A^{-1} \|_\infty = 1/12$ (here $\| \cdot \|_\infty$ stands for the infinity norm of the square matrix). Further, the obvious inequality $| 2n^3\Delta^3 f_{i-1} | \leqslant 8n^3\omega(f, 1/n)$ holds. From the two above inequalities we obtain

$$\max_{1\leqslant i\leqslant n-2} |\gamma_i| \leqslant \frac{2}{3}\, n^3\omega\left(f, \frac{1}{n}\right). \tag{4.2}$$

Some simple connections between the quantities γ_i and $s_i^{(j)}$ were given by Herriot and Reinsch [8], namely,

$$s_i^{\mathrm{IV}} = 60n(\gamma_i - \gamma_{i-1}) \qquad (i = 1, 2,\dots, n-1), \tag{4.3}$$

$$s_i''' = 30(\gamma_i + \gamma_{i-1}) \qquad (i = 1, 2,\dots, n-1), \tag{4.4}$$

$$s_i'' = n^2\Delta^2 f_{i-1} + \frac{1}{2n}(\gamma_{i-2} + 7\gamma_{i-1} - 7\gamma_i - \gamma_{i+1}) \qquad (i = 1, 2,\dots, n-1),$$

$$s_0'' = n^2\Delta^2 f_0 - \frac{1}{2n}(27\gamma_1 + \gamma_2), \tag{4.5}$$

$$s_n'' = n^2\Delta^2 f_{n-2} + \frac{1}{2n}(\gamma_{n-3} + 27\gamma_{n-2}),$$

$$s_i' = \frac{n}{2}(f_{i+1} - f_{i-1}) - \frac{1}{4n^2}(\gamma_{i-2} + 19\gamma_{i-1} + 19\gamma_i + \gamma_{i+1}) \quad (i = 1, 2,..., n-1),$$

$$s_0' = n\Delta f_0 - \frac{n}{2}\Delta^2 f_0 + \frac{1}{4n^2}(25\gamma_1 + \gamma_2), \tag{4.6}$$

$$s_n' = n\Delta f_{n-1} + \frac{n}{2}\Delta^2 f_{n-2} - \frac{1}{4n^2}(\gamma_{n-3} + 25\gamma_{n-2})$$

(we assume here that $\gamma_{-1} = \gamma_0 = \gamma_{n-1} = \gamma_n = 0$). From relations (4.2)–(4.6) the desired inequalities of this lemma follow. ∎

THEOREM 4.1. *For equidistant knots we have* $\| L_n^5 \| = 21/4$.

The proof (in which Lemma 4.1 is used) is quite similar to the that of [17, Theorem 2]. For this reason it is omitted.

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